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A completely integrable Hamiltonian motion on the surface of a sphere

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Abstract. It has been shown (Gaffet B 1996 *J. Fluid. Mech.* **325** 113) that the Hamiltonian in three-dimensional flat space, defined by the simple equations of motion

$$xx''(t) = yy''(t) = zz''(t) = \frac{\text{constant}}{(xyz)^{2/3}}$$

is reducible by transformations of variables, to another Hamiltonian governing the two-dimensional motion of a particle on the surface of a sphere. The equations of motion were shown to possess the Painlevé property and their integration was reduced to essentially, one quadrature.

The earlier analysis essentially concerned the—comparatively much simpler— case where the second integral vanishes; all such solutions were found to be described by elliptic functions. We show in the present work that, when the energy integral (denoted m) and the second integral (denoted ε) are related by

$$8m^3 = (8 + 20\varepsilon - \varepsilon^2) \pm (\varepsilon + 8)\sqrt{\varepsilon(\varepsilon + 8)}$$

the general solution is again represented by elliptic functions. The separation of variables is completed, and the solution presented in detail for the case $m = -3$, chosen as an example.

1. Introduction

The Hamiltonian which is the subject of the present work has its origin in Ovsiannikov (1965) and Dyson's (1968) fluid dynamical model of spinning gas clouds maintaining an ellipsoidal shape. As Dyson noted, his model can equivalently be described as a single-particle Hamiltonian in nine-dimensional space, the parameter space of the (3×3) matrices that represent the instantaneous dilatation, deformation and orientation of the cloud. In cases without rotation, the (3×3) matrices become diagonal and the equivalence is with a single-particle Hamiltonian in three-dimensional flat space. For a polytrope with adiabatic index γ, the corresponding equations of motion assume the simple form

$$xx''(t) = yy''(t) = zz''(t) = \frac{\text{constant}}{(xyz)^{\gamma-1}}. \tag{1.1}$$

Recently it has been shown (Gaffet 1996) that the above system is completely integrable when the cloud's gas is monatomic ($\gamma = 5/3$); moreover, the radial part of the motion may then be separated out, thereby reducing the Hamiltonian to a two-dimensional one, governing the motion of a particle on the surface of the unit sphere (the proof is summarized in the appendix). The subject of the present paper is to study the properties of that Hamiltonian defined on the 2-sphere. The corresponding spherical motions may be derived from the Lagrangian

$$L = \frac{1}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - V_S \tag{1.2}$$

where $d\sigma$ is the element of arc-length on the sphere $x^2+y^2+z^2 = 1$ and V_S is the potential, which in this model is of the form

$$V_S = \frac{3/2}{(xyz)^{2/3}}. \tag{1.3}$$

The momenta π_1 and π_2 are just the covariant velocity components, using the spherical metric. Using coordinates $H \equiv y/x$ and $K \equiv z/x$, the resulting equations of motion read

$$\frac{d}{dt}\left(\frac{\dot{H}}{\delta}\right) = \frac{\delta}{(HK)^{2/3}}\left(\frac{1-H^2}{H}\right) \qquad \frac{d}{dt}\left(\frac{\dot{K}}{\delta}\right) = \frac{\delta}{(HK)^{2/3}}\left(\frac{1-K^2}{K}\right) \tag{1.4}$$

where the dot in $\dot{H}$ and $\dot{K}$ symbolizes differentiation with respect to time t and $\delta \equiv (1 + H^2 + K^2) = 1/x^2$. Remarkably, they possess the Painlevé property—not as functions of time, but *as functions of another independent variable* u

$$u = \int \frac{\delta\, dt}{(HK)^{2/3}}. \tag{1.5}$$

The *dependent* variables require some modification too, it is strictly speaking the functions $U(u)$ and $V(u)$ which have the Painlevé property, where

$$U \equiv H^{2/3} \qquad V \equiv K^{2/3}. \tag{1.6}$$

The new form of the equations of motion is then

$$\frac{d}{du}\left(\frac{U'}{V\sqrt{U}}\right) = \frac{2}{3}\left(\frac{1-U^3}{U^{3/2}}\right) \qquad \frac{d}{du}\left(\frac{V'}{U\sqrt{V}}\right) = \frac{2}{3}\left(\frac{1-V^3}{V^{3/2}}\right) \tag{1.7}$$

where the prime denotes differentiation with respect to u.

There are two first-integrals: the energy constant

$$\frac{1}{2}\left(\frac{d\sigma}{dt}\right)^2 + V_S = \frac{9m}{2} \tag{1.8}$$

or explicitly

$$4m = \left[\frac{(1+V^3)U'^2}{UV^2} - 2U'V' + \frac{(1+U^3)V'^2}{U^2V}\right] + \frac{4(1+U^3+V^3)}{3UV} \tag{1.9}$$

and the second integral I_2

$$I_2 = \frac{3}{4}\frac{U'V'}{UV}\left(\frac{U'}{U} - \frac{V'}{V}\right) + \left[\frac{U'}{V^2}(V^3-1) - \frac{V'}{U^2}(U^3-1)\right]. \tag{1.10}$$

The system is accordingly reducible to a first-order ordinary differential equation (ODE), which must be of a type integrable by quadrature, its integrating factor being determined once the second integral is known. However, the integrating factor obtained $\Psi(U, V)$ has a very complicated form. The fact that the system is completely integrable indicates that any quadrature involved ought to be reducible to quadratures of functions of *one* independent variable only. It is the purpose of this work to show how this separation of variables can be performed, at least in some special cases (a completely general treatment is deferred to a future work). The special cases that we study here turn out to be exactly soluble by elliptic functions.

2. General results

We start by summarizing here some general results, some of which have already been derived in our earlier work. First, it should be pointed out that, by nature, the problem has an underlying symmetry under permutation of the three Cartesian coordinates x, y, z; which makes it very useful introducing a 3-vector notation—even though a full rotational symmetry is absent. Denoting $\boldsymbol{x}$ the position vector $\{x, y, z\}$, we form the cross-product

$$\frac{3}{2}\boldsymbol{X} = \frac{\boldsymbol{x} \wedge \boldsymbol{x}'(u)}{(xyz)^{2/3}} \tag{2.1}$$

where $\boldsymbol{X}$ has components $\{X, Y, Z\}$, in terms of which the equations of motion take the simple form

$$\frac{3}{2}\frac{\mathrm{d}X}{\mathrm{d}u} = \frac{(y^2 - z^2)}{yz} \tag{2.2}$$

together with two more equations deducible by circular permutation. The following properties are worth noting,

$$\boldsymbol{x} \cdot \boldsymbol{X} = 0 = \boldsymbol{x}' \cdot \boldsymbol{X} = \boldsymbol{x} \cdot \boldsymbol{X}'. \tag{2.3}$$

The relation between $\boldsymbol{x}$, $\boldsymbol{X}$ and the original variables U, V, U', V' is

$$x = \frac{1}{\sqrt{\delta}} \qquad \text{where } \delta \equiv 1 + U^3 + V^3$$

$$\frac{y}{x} = U^{3/2} \qquad \frac{z}{x} = V^{3/2} \tag{2.4}$$

$$X = \sqrt{UV}\left(\frac{V'}{V} - \frac{U'}{U}\right) \qquad Y = \frac{-V'}{U\sqrt{V}} \qquad Z = \frac{U'}{V\sqrt{U}}. \tag{2.5}$$

In this new notation the first-integrals allow a simple and compact formulation

$$4m = \boldsymbol{X}^2 + \frac{4/3}{(xyz)^{2/3}} \tag{2.6a}$$

$$I_2 = \frac{3}{4}XYZ - (xyz)^{1/3}\left(\frac{X}{x} + \frac{Y}{y} + \frac{Z}{z}\right) \tag{2.6b}$$

and the definition (1.5) of the independent variable u may be written

$$u = \int \frac{\mathrm{d}t}{(xyz)^{2/3}}. \tag{2.7}$$

2.1. The symmetry generators

We adopt the following convention for symmetry generators: $\bar{\delta}_i F$ denotes the variation produced by a generator (G_i) acting on an arbitrary variable F, when t is kept constant ($\bar{\delta}_i t = 0$), while $\delta_i F$ is the corresponding variation at constant u ($\delta_i u = 0$). Clearly, the two types of variation are related by

$$\bar{\delta} F \equiv \delta F + F'(u)\bar{\delta} u. \tag{2.8}$$

The index i which specifies the generator considered will be occasionally omitted when there is no ambiguity.

Thus the generator (G_0) of time translation has components $\bar{\delta} F = \dot{F}(t)$. A second generator ($G_2$) can be deduced from the existence of the second integral I_2: the general theory of Hamiltonian systems (Landau and Lifshitz 1976, Whittaker 1959) indicates that in the presence of an integral of the motion

$$I[q_1, q_2, p_1, p_2] = \text{constant}$$

(where q_i and p_i are the coordinates and momenta), the system admits the symmetry generator (G) defined by

$$\bar{\delta} q_i = -\frac{\partial I}{\partial p_i} \qquad \bar{\delta} p_i = +\frac{\partial I}{\partial q_i}. \tag{2.9}$$

In the present case the momenta π_1 and π_2 canonically conjugate to the coordinates H and K (already calculated in Gaffet (1996)) are

$$\pi_1 = \frac{3}{2\delta}(Z - KX) \qquad \pi_2 = \frac{3}{2\delta}(HX - Y) \tag{2.10}$$

and, conversely,

$$\tfrac{3}{2}X = H\pi_2 - K\pi_1 \qquad -\tfrac{3}{2}Y = HK\pi_1 + (1 + K^2)\pi_2 \qquad \tfrac{3}{2}Z = (1 + H^2)\pi_1 + HK\pi_2. \tag{2.11}$$

The differential of I_2 is

$$\mathrm{d}I_2 = \mathrm{d}X\left[\frac{3}{4}YZ - \frac{(xyz)^{1/3}}{x}\right] + \frac{(xyz)^{1/3}}{3}\,\mathrm{d}\ln x\left[\frac{2X}{x} - \frac{Y}{y} - \frac{Z}{z}\right]$$
$$+(\text{Terms deduced by circular permutation}) \tag{2.12}$$

from which the second generator may be obtained:

$$-\frac{4}{9}\bar{\delta}x = \frac{X}{2}(yY - zZ) + \frac{2x}{3}\frac{(z^2 - y^2)}{(xyz)^{2/3}}. \tag{2.13}$$

Similar formulae giving $\bar{\delta}y$ and $\bar{\delta}z$ may be found by applying a circular permutation.

However, since u, instead of t, is the independent variable appropriate to the Painlevé expansions, it is preferable to use a constant u formulation of generators. In particular, the Bäcklund transformation relating different solutions (if such a relation is indeed present) is expected to be a transformation leaving u unaltered, and therefore obtainable from a combination of generators at constant u. Since the two types of variation are related by (2.8), that raises the question of whether $\bar{\delta}u$, which is defined by the integral

$$-\frac{3}{2}\bar{\delta}u = \int \bar{\delta}\ln(xyz)\,\mathrm{d}u \tag{2.14}$$

may not in fact turn out to be calculable in finite terms. Remarkably, it does permit an expression in closed form:

$$\bar{\delta}u = -\tfrac{9}{8}XYZ. \tag{2.15}$$

Consequently, in a formulation where u is kept invariable, (G_2) is expressed by the following modified formula

$$-\frac{4}{9}\delta\ln x = \frac{3}{4}XYZ(xyz)^{2/3}\frac{(yZ - zY)}{x} + \frac{X}{2}\frac{(yY - zZ)}{x} + \frac{2(z^2 - y^2)}{3(xyz)^{2/3}} \tag{2.16}$$

(and by the formulae deducible by circular permutation). The variations δU and δV may of course be found from

$$\tfrac{3}{2}\delta\ln U \equiv \delta\ln y - \delta\ln x \qquad \tfrac{3}{2}\delta\ln V \equiv \delta\ln z - \delta\ln x. \tag{2.17}$$

2.2. *The potentials Φ, u, S, t*

For an arbitrary variable F, the variation of F produced by the time translation generator (G_0) is $\bar{\delta}_0 F = \dot{F}(t)$, which may be rewritten $(\partial F/\partial t)|_\Phi$, where Φ denotes a quantity that remains constant along each trajectory. Similarly the variation produced by the second generator (G_2) may be written $\bar{\delta}_2 F = (\partial F/\partial \Phi)|_t$—since, by definition of $\bar{\delta}$ the time t does not vary. More formally, the resulting formula for the differential $\mathrm{d}F$

$$\mathrm{d}F = \bar{\delta}_0 F\,\mathrm{d}t + \bar{\delta}_2 F\,\mathrm{d}\Phi \tag{2.18}$$

does determine an exact differential $\mathrm{d}\Phi$ (a closed 1-form $\mathrm{d} \wedge \mathrm{d}\Phi = 0$) as a consequence of the general properties of generator components $\bar{\delta}_i F$. In particular,

$$\mathrm{d}U = \dot{U}\,\mathrm{d}t + \bar{\delta}_2 U\,\mathrm{d}\Phi \qquad \mathrm{d}V = \dot{V}\,\mathrm{d}t + \bar{\delta}_2 V\,\mathrm{d}\Phi \tag{2.19}$$

and, conversely

$$\mathrm{d}t = [\bar{\delta}_2 V\,\mathrm{d}U - \bar{\delta}_2 U\,\mathrm{d}V]/\det \tag{2.20a}$$

$$\mathrm{d}\Phi = [\dot{U}\,\mathrm{d}V - \dot{V}\,\mathrm{d}U]/\det \tag{2.20b}$$

where $\det \equiv (\dot{U}\bar{\delta}_2 V - \dot{V}\bar{\delta}_2 U)$. Since $(\dot{U}\,\mathrm{d}V - \dot{V}\,\mathrm{d}U) = 0$ (where $\dot{U}$ and $\dot{V}$ are implicit functions of U and V) is just the first-order ODE to which our system may be reduced, the latter equation (2.20b) shows that det is the integrating factor. In addition, it shows that both Φ and the time t may be viewed as potentials defined (up to arbitrary additive constants) on the two-dimensional space $\{U, V\}$, i.e. on the 2-sphere.

The same reasoning using the constant u formulation of generators and the u-translation generator (G_1) as the first generator, in place of (G_0), leads to the conclusion that the variable u, as well as the time t, plays the role of a potential defined on the unit sphere.

There is a fourth potential, whose existence is ensured by the general theory, the abbreviated action S, defined by

$$\mathrm{d}S = (\pi_1\,\mathrm{d}H + \pi_2\,\mathrm{d}K). \tag{2.21}$$

Along trajectories, this reduces to $S = \int L\,\mathrm{d}t + (9m)/2t$, since $9m/2$ is the energy; rewriting the Lagrangian $L = ((9m)/2 - 2V_S)$ and noting that the potential term here is just

$$V_S = \frac{3}{2}\frac{\mathrm{d}u}{\mathrm{d}t} \tag{2.22}$$

we obtain an expression in closed form, $S = 9mt - 3u$, valid along each trajectory. This means that the potential S is a function of the three other potentials, $S = 9mt - 3u + f(\Phi)$. The function $f(\Phi)$ must of course be linear and a detailed calculation gives the complete formula

$$S = 9mt - 3u + \frac{9I_2}{4}\Phi. \tag{2.23}$$

Since the existence of the potentials S, t and Φ is a direct consequence of the general Hamiltonian theory, the above relation may conversely be viewed as a proof that u is also a potential (it constitutes an explanation for the fact that the integral $\bar{\delta}u$ introduced in section 2.1 was found calculable in closed form).

The space considered being bi-dimensional, any differential is a linear combination of, for example, $\mathrm{d}u$ and $\mathrm{d}\Phi$; in particular,

$$\mathrm{d}t = \frac{UV}{\delta}\left[\mathrm{d}u + \frac{9}{8}XYZ\,\mathrm{d}\Phi\right] \tag{2.24}$$

as a direct consequence of (2.15). Then the relation (2.23) entails

$$\mathrm{d}S = 3\,\mathrm{d}u\left[3m\frac{UV}{\delta} - 1\right] + \mathrm{d}\Phi\left[\frac{81m}{8}\frac{UV}{\delta}XYZ + \frac{9}{4}I_2\right]. \tag{2.25}$$

2.3. *The integrating factor*

Writing $\mathrm{d}\Phi = [V\,\mathrm{d}V - \alpha U\,\mathrm{d}U]/\Psi$, where

$$\alpha \equiv \frac{VV'}{UU'} \tag{2.26}$$

the analysis of the preceding section shows that the integrating factor Ψ must be given by

$$\frac{\Psi}{V} \equiv \bar{\delta}_2 V - \frac{V'}{U'}\bar{\delta}_2 U \equiv \delta_2 V - \frac{V'}{U'}\delta_2 U. \tag{2.27}$$

This allows a simple representation in determinant form

$$\Psi \equiv \frac{2}{Z}\sqrt{V\delta}\begin{vmatrix} x, m_X, I_{2X} \\ y, m_Y, I_{2Y} \\ z, m_Z, I_{2Z}\end{vmatrix} \tag{2.28}$$

where $m_X \equiv \partial m/\partial X$ etc, the partial derivatives being taken at x, y and z constant. It will be useful to consider, in addition to α, the related parameter β

$$\beta \equiv \frac{UV^2}{U'}\left(\frac{U'}{U} - \frac{V'}{V}\right) \equiv (V^2 - \alpha U^2). \tag{2.29}$$

The integrating factor may then be written

$$\Psi \equiv N_\alpha - U^2 N_\beta + 3I_2\frac{UV}{U'}(\beta U - \alpha) \tag{2.30}$$

where

$$N(\alpha,\beta) \equiv (\alpha^3 + \beta^3 + 3m\alpha\beta - 1) \qquad N_\alpha \equiv \frac{\partial N}{\partial \alpha} \qquad N_\beta \equiv \frac{\partial N}{\partial \beta} \tag{2.31}$$

and the function $\alpha(U, V)$ is implicitly defined by the algebraic equation

$$(3mUV - \delta)N^2 = \frac{3}{4}I_2^2(\alpha^2 U + \beta^2 + V)^3. \tag{2.32}$$

Thus Ψ is a quite complicated integrating factor.

2.4. *The Painlevé expansion*

As mentioned in the introduction, our mechanical system possesses the Painlevé property (a summary of Painlevé's method of analysis may be found in Ince (1956)) and this fact alone indicates that it is a completely integrable Hamiltonian system, according to a well known conjecture (Ablowitz and Segur 1977).

The Painlevé analysis is based on a consideration of the Taylor series expansion of the general solution in the vicinity of its movable singular points. The only singularities present in the functions $U(u)$ and $V(u)$ are poles, of order one or two. The *generic* type of singularity (involving four integration constants) is the simple pole

$$U = \frac{a_0}{u} + a_1 + a_2 u + \cdots \qquad V = \frac{b_0}{u} + b_1 + b_2 u + \cdots \tag{2.33}$$

where the a_i's and b_i's are symmetrically related as

$$a_0 b_0 = -3/4 \qquad a_0 b_1 = a_1 b_0 \qquad (a_0 b_2 - 2a_1 b_1 + a_2 b_0) = 0 \tag{2.34}$$

and the fourth integration constant provided by the arbitrary translations of the pole is implicit.

The variables α and β tend at the pole to finite limits $\hat{\alpha}$ and $\hat{\beta}$, respectively,

$$\hat{\alpha}^{1/4} = \frac{\sqrt{3}}{2a_0} \qquad \hat{\beta} = \frac{9}{4a_0^2}(a_1^2 - a_0 a_2). \tag{2.35}$$

When m and I_2 are fixed, these two constants cannot be independent. We find that $\hat{\alpha}$ and $\hat{\beta}$ are algebraically related by

$$N(\hat{\alpha}, \hat{\beta}) \equiv (\hat{\alpha}^3 + \hat{\beta}^3 + 3m\hat{\alpha}\hat{\beta} - 1) = \frac{\sqrt{3}}{2} I_2 \hat{\alpha}^{3/4}(\hat{\alpha}^{3/2} - 1). \tag{2.36}$$

For each pair of values of m and I_2, the algebraic nature of the above relation strongly constrains the analytical form of the solutions. Thus, when $I_2 = 0$, the relation (2.36), which is then cubic, is of genus one and hence parametrable by elliptic functions; the corresponding general solution too is represented by elliptic functions.

Let us introduce in place of $\hat{\alpha}$ and $\hat{\beta}$ two new constants λ and μ

$$\lambda = \frac{(\hat{\alpha}^{3/2} - 1)}{2\hat{\beta}\hat{\alpha}^{1/4}} \qquad \mu = 2\lambda^3 + \frac{(\hat{\alpha}^{3/2} + 1)}{2\hat{\alpha}^{3/4}} \tag{2.37}$$

so the relation (2.36) becomes

$$\mu^2 = [4\lambda^6 + \lambda^3 I_2 \sqrt{3} - 3m\lambda^2 + 1]. \tag{2.38}$$

Clearly, it is of genus one (parametrable by elliptic functions) whenever the sixth-degree polynomial on the right-hand side has a double root, a condition which is realized when m and I_2 satisfy the relation

$$8m^3 = (8 + 20\varepsilon - \varepsilon^2) \pm (\varepsilon + 8)\sqrt{\varepsilon(\varepsilon + 8)} \qquad \varepsilon = \frac{3}{16} I_2^2. \tag{2.39}$$

For such values of m and I_2, the analytical form of the solutions must be simpler than in general, and is not expected to involve functions of a more complicated type than elliptic. We shall treat in the next section as a typical example the case where

$$m = -3 \text{ and } I_2 = 8i\sqrt{2/3}.$$

2.5. *Rescaling*

The results of the preceding sections all refer to the Hamiltonian

$$H = \frac{1}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 + V_S \qquad \text{where} \qquad V_s = \frac{3(1 + U^3 + V^3)}{2UV}.$$

The absolute scale of the potential term V_S is, obviously, a reducible parameter; the product $V_S t^2$ need only be kept unaltered. Changing *the sign* of the potential however, entails rescaling t (and u and I_2) by the imaginary factor i. Conversely a solution corresponding to a pure imaginary value of I_2 (and pure imaginary values of u) may be re-interpreted as a *real solution* of the modified Hamiltonian

$$H = \frac{1}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - V_S.$$

(The fact that u must take imaginary values in order for this re-interpretation to be valid is not a difficulty here, since the functions of u that are considered in the present work are all analytical functions, defined throughout the complex u-plane.)

3. The case where $m = -3$

We now turn to consideration of the case where $m = -3$ and $I_2^2 = -128/3$.

3.1. The parametrization problem

The main difficulty lies in obtaining an appropriate parametrization of the implicit relations defining the two functions $U'(U, V)$ and $V'(U, V)$, or equivalently, $\boldsymbol{X}(U, V)$. Let us first write down these relations in a form as convenient as possible. We introduce new variables ξ, η and ζ proportional to X/x, Y/y, Z/z, through the relations

$$X/x = -\frac{I_2}{4}\xi\sqrt{\frac{\delta}{UV}} \qquad Y/y = -\frac{I_2}{4}\eta\sqrt{\frac{\delta}{UV}} \qquad Z/z = -\frac{I_2}{4}\zeta\sqrt{\frac{\delta}{UV}} \tag{3.1}$$

which may also be written

$$\zeta = -\frac{4}{I_2}\frac{U'}{V^2} \tag{3.2a}$$

$$\eta = +\frac{4}{I_2}\frac{V'}{U^2} \tag{3.2b}$$

$$\xi = -(\eta U^3 + \zeta V^3) \qquad \text{(from } \boldsymbol{x}\cdot\boldsymbol{X} = 0\text{)}. \tag{3.2c}$$

Then we obtain

$$\boldsymbol{X}^2 = \frac{-8}{3UV}[\xi^2 + \eta^2 U^3 + \zeta^2 V^3] \tag{3.3}$$

$$(\delta - 3mUV) = (1 + U^3 + V^3 + 9UV) \tag{3.4}$$

so the integral of energy takes the form

$$(1 + U^3 + V^3 + 9UV) = 2[\xi^2 + \eta^2 U^3 + \zeta^2 V^3] \tag{3.5}$$

while the second integral simply reads

$$2\xi\eta\zeta + (\xi + \eta + \zeta) = 4 \tag{3.6}$$

and ξ, η and ζ also satisfy the relation (3.2c)

$$(\xi + \eta U^3 + \zeta V^3) = 0. \tag{3.7}$$

That is the set of algebraic equations (three relations (3.5–7) between five variables ξ, η, ζ, U and V) that have to be solved parametrically. As a first step we choose η and $\theta \equiv (2\xi\zeta + 1)$ as parameters; then equation (3.6) gives $(\xi + \zeta)$; ξ and ζ are accordingly determined by an equation of the second degree

$$\xi^2 + \xi(\eta\theta - 4) + \frac{(\theta - 1)}{2} = 0. \tag{3.8}$$

It has discriminant Δ

$$\Delta = (\eta\theta - 4)^2 + 2(1 - \theta) \tag{3.9}$$

and its two roots are expressed by

$$2\xi = (4 - \eta\theta) + \sqrt{\Delta} \qquad 2\zeta = (4 - \eta\theta) - \sqrt{\Delta}. \tag{3.10}$$

Similarly, introducing the new variable

$$\pi \equiv \frac{V}{U^2} \tag{3.11}$$

equation (3.7) takes the form of a second degree equation for U^3

$$\zeta\pi^3 U^6 + \eta U^3 + \xi = 0. \tag{3.12}$$

Its discriminant is

$$D = \eta^2 + 2\pi^3(1-\theta) \tag{3.13}$$

and U and V are accordingly given by

$$U^3 = \frac{(\sqrt{D}-\eta)}{2\zeta\pi^3} \qquad V = \pi U^2. \tag{3.14}$$

In this way the variables ξ, ζ, U and V may be eliminated, and the algebraic system (3.5–7) is thereby converted into a single algebraic relation involving π, η and θ only,

$$\frac{[9\pi(\theta-1)+\varphi(\eta)]}{\theta} + (1+4\eta-\eta^2\theta) + \sqrt{D\Delta} = 0 \tag{3.15}$$

where

$$\varphi(\eta) \equiv (2\eta^2 - 8\eta - 1). \tag{3.16}$$

Taking the square of the above relation (3.15), so as to eliminate the square root, changes it into a cubic equation implicitly defining $\pi(\eta,\theta)$. The corresponding algebraic surface in the space of coordinates $\{\pi,\eta,\theta\}$ presents *two singular lines* the locus of double points of the plane sections; the first one is the curve

$$\theta = 0 \qquad \pi = \varphi(\eta)/9 \tag{3.17}$$

and the second one is

$$3\pi = \psi(\eta) \equiv (2\eta^2 - 2\eta - 1) \qquad \theta = (2\eta-1)/\pi. \tag{3.18}$$

The sections $\eta =$ constant are quartic curves in coordinates $\{1/\pi,\theta\}$; the presence of a pair of double points being ascertained, these curves are of genus one and thus admit an elliptic parametrization. Further study shows that the radical involved may be rationalized (by relaxing the assumption of η constant), and we are thus led to the following fully rational representation of the surface in $\{\pi,\eta,\theta\}$ space

$$-2\eta = \frac{E_1}{E_2} \tag{3.19}$$

where

$$E_1 \equiv \sigma^2\lambda(3\lambda-2) + 2\sigma(2\lambda^2-3\lambda+2) - (\lambda+2)^2$$
$$E_2 \equiv \sigma^2(4\lambda^2+\lambda-2) + 2\sigma\lambda(3\lambda-2) + (2\lambda^2-3\lambda+2) \tag{3.20}$$
$$\theta = -2E_2\frac{T_1}{T_2} \tag{3.21}$$

where

$$T_1 \equiv \sigma^2(23\lambda^2-17\lambda+2) + 2\sigma(13\lambda^2-25\lambda+10) + (5\lambda^2-23\lambda+14)$$
$$\begin{aligned} T_2 \equiv{}& \sigma^4(3\lambda-2)^2(\lambda^2-4\lambda+2) - 4\sigma^3(3\lambda-2)(\lambda^3+9\lambda^2-9\lambda+2)\\ &-2\sigma^2(25\lambda^4+22\lambda^3-66\lambda^2+40\lambda-8)\\ &-4\sigma(11\lambda^4-7\lambda^3+\lambda^2-8\lambda+4)\\ &-(11\lambda^4-20\lambda^3+18\lambda^2-40\lambda+24) \end{aligned} \tag{3.22}$$

and

$$\pi = \frac{\lambda T_2}{2(3\lambda-2)E_2^2}. \tag{3.23}$$

The simple expression of the product $\pi\theta$ (which does not involve T_2) is worth noting:

$$\pi\theta = \frac{-\lambda T_1}{(3\lambda - 2)E_2}. \tag{3.24}$$

The above result (i.e. the representation 3.19–24) is essential, and the integration of the equations of motion, completed in the next section, is thus made a comparatively simple task.

3.2. The differential system for $\lambda(u)$ and $\sigma(u)$

The next step is to obtain the differential equations satisfied by the two parameters λ and σ as functions of the independent variable u. Since the functions $\eta(\lambda, \sigma)$ and $\theta(\lambda, \sigma)$ are given, this is achieved by computing $\eta'(u)$ and $\theta'(u)$ and resolving the resulting linear system for $\lambda'(u)$ and $\sigma'(u)$:

$$\frac{\partial\eta}{\partial\lambda}\lambda'(u) + \frac{\partial\eta}{\partial\sigma}\sigma'(u) = \eta'(u) \qquad \frac{\partial\theta}{\partial\lambda}\lambda'(u) + \frac{\partial\theta}{\partial\sigma}\sigma'(u) = \theta'(u). \tag{3.25}$$

The coefficients of this linear system are readily obtainable by differentiation of the formulae (3.19–22) found in the preceding section; the right-hand sides are given by

$$\eta'(u) = \frac{I_2}{16\pi}\left[2\eta\left(\eta + 2\zeta\frac{V^3}{U^3}\right) + \frac{(V^3 - 1)}{U^3}\right] \qquad \theta'(u) = \frac{-I_2}{4\pi}\left[\theta\zeta\frac{V^3}{U^3} + 2 - \zeta\right]. \tag{3.26}$$

We also note that

$$\pi'(u) = \frac{I_2}{4}\left[\eta + 2\zeta\frac{V^3}{U^3}\right]. \tag{3.27}$$

Eliminating the dependence on variables U, V, ξ and ζ, this may be rewritten

$$\begin{aligned}
\eta'(u) &= \frac{I_2}{16\pi(\theta - 1)}\{[\eta\theta - 2(\eta - 2)]\sqrt{D} - \eta\sqrt{\Delta}\} \\
&= \frac{24}{I_2\sqrt{\Delta}}\left\{\left[\eta - \frac{2(\eta - 2)}{\theta}\right] + \frac{\varphi(\eta)}{9\pi}\left[\eta + \frac{2(\eta - 2)}{\theta}\right]\right\} \\
\theta'(u) &= \frac{16}{3I_2\pi}(\sqrt{\Delta} + \theta\sqrt{D}) = \frac{16}{3I_2\pi\sqrt{\Delta}}[9\pi(1 - \theta) - \varphi(\eta) + 2\theta^2\eta^2 - 3\theta(4\eta + 1) + 18] \\
\pi'(u) &= \frac{I_2}{4}\sqrt{D}.
\end{aligned} \tag{3.28}$$

Concerning the remarkably symmetric expression of $\eta'(u)$, the following identities are worth noting

$$[\eta\theta - 2(\eta - 2)] \equiv \sqrt{\Delta + 2\varphi(1 - \theta)} \tag{3.29a}$$

$$\eta[\eta\theta - 2(\eta - 2)] \equiv \sqrt{D\Delta} + (9\pi - \varphi)\frac{(\theta - 1)}{\theta} \tag{3.29b}$$

(the latter equation (3.29*b*) constituting an alternative form of the algebraic relation (3.17) linking π, η and θ).

Let us introduce a function $\mu(\lambda)$, through

$$\mu^2 \equiv (3\lambda - 2)\nu(\lambda) \qquad \nu(\lambda) \equiv (\lambda^3 + 3\lambda - 2). \tag{3.30}$$

The essential property of the radicals is that $\sqrt{\Delta}/\mu$ and $\sqrt{D}/\mu$ are both rational functions of the two parameters λ and σ.

The resulting differential system satisfied by $\lambda(u)$ and $\sigma(u)$, in spite of the complexity of the transformation of variables involved, has a simple form

$$\lambda'(u) = \frac{16}{3I_2}\mu(\lambda) \tag{3.31a}$$

$$\sigma'(u) = \frac{16}{3I_2}\frac{(\sigma-1)}{\lambda\mu}[\sigma(2\lambda-1)(3\lambda-2)+(\lambda^3+3\lambda^2-5\lambda+2)]. \tag{3.31b}$$

We observe that the separation of variables is completed, since equation (3.31a) for $\lambda(u)$ does not involve the other function σ. In addition, the equation for $\sigma(\lambda)$ is of Riccati type, and is linearized by the transformation $y \equiv 1/(\sigma-1)$.

Integration of the Riccati equation yields the general solution of the system

$$3\Phi = \frac{\lambda(3\lambda-2)}{\mu(\sigma-1)} + \int \frac{(2\lambda-1)(3\lambda-2)}{\nu(\lambda)}\frac{\mathrm{d}\lambda}{\mu} \tag{3.32}$$

where Φ is the integration constant, normalized here in such a way as to precisely coincide with the potential Φ described in the preceding sections. The second potential u, in view of (3.31a), must have the form

$$u = \frac{3I_2}{16}\int\frac{\mathrm{d}\lambda}{\mu} + g(\Phi). \tag{3.33}$$

Compatibility with the general formula

$$\mathrm{d}u = \frac{V}{\Psi U'}(\delta V\,\mathrm{d}U - \delta U\,\mathrm{d}V) \tag{3.34}$$

(which is the analogue for du of equation (2.20) for dt), together with the expressions (2.16) and (2.17) of the second generator, yields the complete formula

$$u = \frac{3I_2}{16}\left[\int\frac{\mathrm{d}\lambda}{\mu} - 5\Phi\right]. \tag{3.35}$$

3.3. *The action*

To sum up, the general solution is found through the following steps: first, an (elliptic) function $\lambda(u)$ is obtained through the quadrature (3.35); then the function $\sigma(u)$ is found through the quadrature in (3.32)—which in the present case is an elliptic integral; to complete the solution, one should then calculate the time t or, equivalently, the action S, which is related to t by (2.23).

In differential form, the action is given by

$$\mathrm{d}S = \frac{9UU'}{4V\zeta\delta}[(\zeta-\xi)\,\mathrm{d}\ln U + (\xi-\eta)\,\mathrm{d}\ln V] \tag{3.36}$$

but, applying the formula (2.25), we more directly obtain

$$\frac{\partial S}{\partial\sigma} = \frac{81}{16}I_2\frac{\partial\Phi}{\partial\sigma}\left[1+\frac{UV}{\delta}\left(5-\frac{6XYZ}{I_2}\right)\right] \tag{3.37}$$

together with a similar result for $(\partial S/\partial\lambda)$. $(\partial\Phi/\partial\sigma)$ is readily deduced from equation (3.32),

$$XYZ \equiv \frac{2I_2}{3}\xi\eta\zeta \equiv \frac{I_2}{3}\eta(\theta-1)$$

and

$$\frac{(1+U^3+V^3)}{UV} \equiv \frac{1}{\pi}\left[(2\eta^2+1)+\frac{(\varphi(\eta)-9\pi)}{\theta}\right]. \tag{3.38}$$

Substituting the parametric representation (3.19–23), one finds

$$\frac{(9\pi - \varphi(\eta))}{\theta} = \frac{T_2 T_3}{2(3\lambda - 2)E_2^3} \tag{3.39}$$

where

$$T_3 \equiv \sigma^2(3\lambda - 2)(\lambda^2 + 2\lambda - 2) + 2\sigma(3\lambda - 2)(2\lambda^2 + \lambda + 2) + (3\lambda^3 - 2\lambda^2 + 2\lambda + 4) \tag{3.40}$$

and then

$$\frac{\delta}{UV} = \frac{\Sigma_6(\lambda;\sigma)}{E_2 T_2} \tag{3.41}$$

where Σ_6 is a polynomial of degree six in σ (and also in λ), of which we shall merely write down the leading term,

$$\Sigma_6 \equiv \sigma^6(3\lambda - 2)(5\lambda - 3)(31\lambda^4 + 36\lambda^3 - 6\lambda^2 - 56\lambda + 24) + \cdots. \tag{3.42}$$

Then the coefficient appearing in (3.37) is found as

$$\left[1 + \frac{UV}{\delta}\left(5 - \frac{6XYZ}{I_2}\right)\right] = 2(\sigma - 1)^2 \nu(\lambda)\frac{\Sigma_4}{\Sigma_6} \tag{3.43}$$

where Σ_4 is another polynomial, quartic in σ,

$$\Sigma_4 \equiv \sigma^4(3\lambda - 2)(11\lambda^2 - 20\lambda + 8) + \cdots. \tag{3.44}$$

Finally, the action is given by the quadrature

$$S = -\frac{27 I_2}{8}\lambda\mu \int \frac{\Sigma_4}{\Sigma_6}\mathrm{d}\sigma. \tag{3.45}$$

The six residues are found to have identical values, up to sign

$$\frac{\Sigma_6'(\sigma)}{\Sigma_4} = \pm 6\mathrm{i}\sqrt{2}\lambda\mu \qquad \text{whenever } \Sigma_6 = 0 \tag{3.46}$$

a result which entails decomposability of Σ_6

$$\Sigma_6 \propto (A^2 + 2\mu^2 B^2)$$

where A and B are polynomials cubic in σ.

Σ_6 and Σ_4 turn out to be expressible in the form

$$\Sigma_6 \equiv \frac{(A^2 + 2\mu^2 B^2)}{(3\lambda - 2)(5\lambda - 3)} \qquad \Sigma_4 \equiv \frac{(A\partial B/\partial\sigma - B\partial A/\partial\sigma)}{3\lambda(3\lambda - 2)(5\lambda - 3)} \tag{3.47}$$

where

$$\begin{aligned}\frac{A}{(3\lambda - 2)} &\equiv [\sigma^3(5\lambda - 3)(5\lambda^2 + 4\lambda - 4) + 3\sigma^2(3\lambda - 2)(7\lambda^2 - \lambda - 2)\\ &\quad + 3\sigma(3\lambda - 2)(5\lambda^2 - 5\lambda + 2) + (11\lambda^3 - 23\lambda^2 + 32\lambda - 12)]\\ B &\equiv [\sigma^3(3\lambda - 2)(5\lambda - 3) + 3\sigma^2(3\lambda - 2)(2\lambda - 1) + 3\sigma(5\lambda^2 + \lambda - 2)\\ &\quad + (6\lambda^2 + \lambda - 6)].\end{aligned} \tag{3.48}$$

The action is accordingly obtained in closed form

$$S = \frac{-9I_2}{16}\left\{\sqrt{2}\tan^{-1}\left(\mu\sqrt{2}\frac{B}{A}\right) + h(\lambda)\right\} \tag{3.49}$$

and the integration 'constant' $h(\lambda)$, which may be determined from the other partial derivative $\partial S/\partial\lambda$, is the elliptic integral

$$h(\lambda) \equiv \int \frac{(3\lambda - 2)}{(5\lambda - 3)}\frac{\mathrm{d}\lambda}{\mu}. \tag{3.50}$$

3.4. The curves $\lambda = constant$

The parametrization obtained in section 3.1 determines all the physical variables in terms of λ and σ. Conversely, it would be of interest to have an explicit expression of the parameters (particularly of λ, which plays an essential role) directly in terms of physical variables π, η and θ. We obtain the following formula:

$$\left(\frac{3\lambda-2}{\lambda}\right)\pi\theta = \frac{3(3\pi-\psi)}{(\pi\theta+1-2\eta)} - 2(\eta+1) = \frac{(9\pi-\varphi)-2\pi\theta(\eta+1)}{(\pi\theta+1-2\eta)}. \tag{3.51}$$

The curves of constant λ are then the intersections of surfaces represented by equations (3.15) and (3.51); they are manifestly *conic sections* (ellipses of hyperbolae) *in the* (η, ρ) *plane*, letting

$$\rho \equiv \pi\theta \tag{3.52}$$

for conciseness. It is worth noting that, as a consequence of (3.51), π is a rational function of the conic curve's coordinates (η, ρ).

Each conic is tangential to the second singularity line, which is (in the (η, ρ) plane) the straight line $(\rho + 1 - 2\eta) = 0$, at the point

$$\eta = \frac{(2-\lambda)}{2(2\lambda-1)} \qquad \rho = \frac{3(1-\lambda)}{(2\lambda-1)}$$

and each intersects the first singularity line, which is the straight line $\rho = 0$, at two points. The equation of the conic may be written

$$X_1^2 + aX_1X_2 + bX_2^2 + cX_1 = 0 \tag{3.53}$$

where

$$X_1 \equiv (\rho + 1 - 2\eta) \qquad X_2 \equiv \rho + \frac{3(\lambda-1)}{(2\lambda-1)} \tag{3.54}$$

$$a = \frac{-2(\lambda-1)(\lambda^2+6\lambda-4)}{\lambda(\lambda^2-4\lambda+2)} \qquad b = \frac{-2(2\lambda-1)^2}{(\lambda^2-4\lambda+2)} \qquad c = \frac{-6\mu^2}{\lambda(2\lambda-1)(\lambda^2-4\lambda+2)}. \tag{3.55}$$

3.5. Effect on λ *of permutations of the coordinate axes*

The formula (3.31*a*) suggests that λ might be invariant under permutations of the Cartesian coordinates (x, y, z) since du is manifestly invariant. Closer inspection reveals that, since u itself need not necessarily be invariant, λ might instead be transformed into another value λ^*, related to λ by a fixed translation of the independent variable u

$$\lambda^* = \lambda(u + h). \tag{3.56}$$

This is a point worth investigating (in view of the importance of the role played by permutation invariant quantities, such as the potential $V_s = \frac{3}{2}(xyz)^{-2/3}$, in the present problem).

Let us consider for definiteness the permutation of coordinates x and y, leaving z unaltered. Clearly, it induces a corresponding permutation of ξ and η, without affecting ζ. The transformed values π^*, η^* and θ^* of the physical variables may be deduced, and the new value λ^* of λ may be found by application of the formula (3.51). The result is, as indicated by the general form of equation (3.56), a function of λ alone, it does not involve the second parameter σ and it is one of the solutions of the second degree equation

$$(13\lambda^2 - 18\lambda + 6)\lambda^{*2} - 2\lambda^*(3\lambda-1)(3\lambda-2) + 2\lambda(3\lambda-2) = 0. \tag{3.57}$$

We conclude that λ is *not* in fact permutation invariant. Rather, the invariant quantity is

$$L \equiv \frac{(13\lambda^2 - 18\lambda + 6)}{\lambda^2(3\lambda - 2)}. \tag{3.58}$$

According to (3.56) λ and λ^* ought to be related by a fixed translation h of the variable u. To find the amount of the translation, it will be convenient to reduce the elliptic function $\lambda(u)$ to its canonical Weierstrass form. Introducing W,

$$W \equiv \frac{(13\lambda - 6)}{2(2 - 3\lambda)} \tag{3.59}$$

we obtain

$$\frac{W'^2(u)}{(4W^3 - 51W - 71)} = \frac{-\lambda'^2(u)}{2\mu^2} = \frac{1}{3} \tag{3.60}$$

showing that $W[v \equiv u/\sqrt{3}]$ is a Weierstrass function. The transformation formula induced by (3.57) on W, reads

$$-2W^* = \frac{(12W^2 + 84W + 131) + 8\mathrm{i}\sqrt{2}W'}{(2W + 3)^2} \tag{3.61}$$

which is precisely (Goursat 1949) the well known translation formula for Weierstrass functions, $v \to v + h$, in the particular case where $W(h) = -3/2$.

We remark that, by application of the doubling formula (Goursat 1949) $W(2h) = -3/2 = W(h)$, meaning that $3h$ *is one of the periods of the elliptic functions considered.*

4. Conclusion

The spherical motion in a potential $V_S \propto (xyz)^{-2/3}$ was shown in a recent work (Gaffet 1996) to present a second integral of the motion and, as a result, to be integrable by quadrature. The integrating factor, however, was a very complicated, implicitly determined function of the coordinates U and V on the sphere. The aim of the present work was to reduce the quadrature involved to its simplest form. In view of the complexity of the calculations involved, it has not been possible here to perform this reduction in the most general case, rather, we have selected the special case, brought to light by a Painlevé analysis, where the two integrals m and $\varepsilon = (3/16)I_2^2$ are related by

$$8m^3 = (8 + 20\varepsilon - \varepsilon^2) \pm (\varepsilon + 8)\sqrt{\varepsilon(\varepsilon + 8)}$$

and we have treated as a typical example the case where $m = -3$. (The zero energy case where $m = 0$ and $I_2 = 4/\sqrt{3}$ is considerably simpler to handle than cases where $m \neq 0$, but we have chosen not to report it here in order not to confuse the reader by a plethora of notation; the case of $m = 1$ and $I_2 = 0$ was treated along with all cases in which I_2 vanishes in Gaffet (1996).)

We have been able to reduce this system to the very simple type

$$\lambda'(u) = \text{ constant } \times \mu(\lambda)$$

where μ^2 is a quartic polynomial in λ, thus not only completing the separation of variables but also obtaining an explicit formulation of the general solution in terms of elliptic functions. We remark that the transformation of variables (section 3.1) leading to this simple result is of surprising complexity.

We hope to complete this study by extending it in a future work to fully arbitrary values of the two integrals of the motion.

Appendix. The separability of the radial motion

We have mentioned in the introduction the Hamiltonian motion in three-dimensional Euclidean space, governed by the equations

$$xx''(\tau) = yy''(\tau) = zz''(\tau) = \frac{1}{(xyz)^{2/3}}. \tag{A.1}$$

We summarize here the proof (given in Gaffet (1996)) that the radial part of the motion separates out and that the projected motion on the unit sphere is again Hamiltonian, although not with respect to the original time coordinate τ.

First, we observe that, as noted by Anisimov and Lysikov (1970), equation (A.1) presents three constants of the motion, denoted E, Σ and E^*, which can be defined through the following single formula:

$$\frac{r^2}{2} \equiv \frac{(x^2 + y^2 + z^2)}{2} = (\tau^2 E + \tau\Sigma + E^*). \tag{A.2}$$

Let us write equations (A.1) in the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(x\frac{\mathrm{d}y}{\mathrm{d}\tau} - y\frac{\mathrm{d}x}{\mathrm{d}\tau}\right) = \left(\frac{x}{y} - \frac{y}{x}\right) \Big/ (xyz)^{2/3} \tag{A.3}$$

using coordinates $H \equiv y/x$ and $K \equiv z/x$ on the unit sphere, and $\delta \equiv (1 + H^2 + K^2) \equiv (r^2/x^2)$, we rewrite

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{r^2}{\delta}\frac{\mathrm{d}H}{\mathrm{d}\tau}\right) = \frac{(1 - H^2)}{H(xyz)^{2/3}}. \tag{A.4}$$

Introducing a new time coordinate t

$$t = \int \frac{\mathrm{d}\tau}{r^2} \tag{A.5}$$

equation (A.4) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{H}}{\delta}\right) = \frac{(1 - H^2)}{H(xyz)^{2/3}} \tag{A.6}$$

where the dot means derivation with respect to t, and the point (x, y, z) is now constrained to lie on the unit sphere; that is to say, $(xyz)^{2/3}$ means $(HK)^{2/3}/\delta$. Another equation obtained by exchanging the roles of H and K, also holds,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{K}}{\delta}\right) = \frac{(1 - K^2)}{K(xyz)^{2/3}}. \tag{A.7}$$

It was shown in our earlier work that (A.6) and (A.7) are the equations governing the motion of a particle of unit mass on the sphere $x^2 + y^2 + z^2 = 1$, in a potential V_S

$$V_S = \frac{3/2}{(xyz)^{2/3}}. \tag{A.8}$$

The energy constant of this spherical motion is the following combination of constants

$$\frac{1}{2}(4EE^* - \Sigma^2) = \frac{1}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 + V_S \tag{A.9}$$

where $\mathrm{d}\sigma^2$ is the line element on the sphere (cf equation 1.8).

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